Midterm Exam — Partial Differential Equations (WBMA008-05)

Wednesday 15 May 2024, 18.30–20.30h

University of Groningen

Instructions

1. The use of calculators, books, or notes is not allowed.
2. All answers need to be accompanied with an explanation or a calculation: only answering "yes", "no", or "42" is not sufficient.
3. If p is the number of marks then the midterm grade is $G = 1 + p/5$.

Problem 1 (5 + 5 + 5 = 15 points)

Consider the following nonuniform transport equation:

$$\frac{\partial u}{\partial t} + x^2 \frac{\partial u}{\partial x} = 0, \quad u(0,x) = \cos(\pi x).$$

(a) Compute all characteristic curves.

(b) Compute the value of the solution u at the point $(t,x) = (1,1)$.

(c) Is the solution u at the point $(t,x) = (1,-2)$ determined by the initial condition?

Problem 2 (9 + 3 + 3 = 15 points)

Consider the function $f : [-\pi, \pi] \to \mathbb{R}$ given by $f(x) = |x| + x$.

(a) Compute the coefficients a_k and b_k of the real Fourier series of f.

(b) Compute the value of the Fourier series at $x = \pi$.

(c) Does the Fourier series of f converge uniformly on $\mathbb{R}$?

Problem 3 (15 points)

Consider the following heat equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u(t,0) = 0, \quad \frac{\partial u}{\partial x}(t,1) - u(t,1) = 0, \quad u(0,x) = f(x).$$

Solve this problem using the ansatz $u(t,x) = e^{\lambda t} v(x)$. Consider all cases for the sign of λ.

Express the final solution as an infinite series and give expressions for the coefficients in terms of the function f.

End of test (45 points)

Solution of problem 1 (5 + 5 + 5 = 15 points)

(a) The characteristic curves $t \mapsto (t, x(t))$ are found by solving the following ordinary differential equation:

$$\frac{dx}{dt} = x^2.$$

(1 point)

Note that the curve $t \mapsto (t, 0)$, i.e. the line $x = 0$, is a characteristic curve.
(1 point)

To find the remaining characteristic curves, we use separation of variables:

$$\int \frac{1}{x^2}\,dx = \int dt \quad \Rightarrow \quad -\frac{1}{x} = t + k \quad \text{or} \quad x = -\frac{1}{t+k}.$$

(3 points)

(b) The point $(t, x) = (1, 1)$ lies on the characteristic curve given for $k = -2$.
(2 point)

This characteristic curve intersects the x-axis in the point $(0, \frac{1}{2})$.
(1 point)

Since the points $(1, 1)$ and $(0, \frac{1}{2})$ lie on the same characteristic curve and the solution u is constant along such a curve, we have

$$u(1, 1) = u(0, \tfrac{1}{2}) = \cos(\tfrac{1}{2}\pi) = 0.$$

(2 points)

(c) The point $(t, x) = (1, -2)$ lies on the characteristic curve given for $k = -\frac{1}{2}$.
(1 point)

Note that the equation

$$x = -\frac{1}{t - 1/2}$$

actually specifies *two distinct curves* in the (t, x)-plane, namely one branch for $t > 1/2$ and another branch for $t < 1/2$. The branch that contains the point $(1, -2)$ does not intersect the x-axis. Therefore, the solution at the point $(t, x) = (1, -2)$ is not determined by the initial condition.
(4 points)

Note: the solution is given by

$$u(t, x) = f(x/(1 + tx)).$$

Solution of problem 2 (9 + 3 + 3 = 15 points)

(a) Note that we can write the given function as follows:

$$f(x) = \begin{cases} 2x & \text{if } 0 \le x \le \pi, \\ 0 & \text{if } -\pi \le x < 0. \end{cases}$$

For $k = 0$ we obtain the coefficient

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} 2x\,dx = \frac{1}{\pi}\left[x^2\right]_0^{\pi} = \pi.$$

(1 point)

For $k \ge 1$ the coefficients a_k are given by

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx \\ &= \frac{1}{\pi}\int_0^{\pi} 2x\cos(kx)\,dx \\ &= \frac{2}{\pi}\left(\left[\frac{x}{k}\sin(kx)\right]_0^{\pi} - \frac{1}{k}\int_0^{\pi}\sin(kx)\,dx\right) \\ &= \frac{2}{\pi}\left(\left[\frac{x}{k}\sin(kx)\right]_0^{\pi} - \frac{1}{k}\left[-\frac{1}{k}\cos(kx)\right]_0^{\pi}\right) \\ &= \frac{2}{k^2\pi}((-1)^k - 1). \end{aligned}$$

(4 points)

For $k \ge 1$ the coefficients b_k are given by

$$\begin{aligned} b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx \\ &= \frac{1}{\pi}\int_0^{\pi} 2x\sin(kx)\,dx \\ &= \frac{2}{\pi}\left(\left[-\frac{x}{k}\cos(kx)\right]_0^{\pi} + \frac{1}{k}\int_0^{\pi}\cos(kx)\,dx\right) \\ &= \frac{2}{\pi}\left(\left[-\frac{x}{k}\cos(kx)\right]_0^{\pi} + \frac{1}{k}\left[\frac{1}{k}\sin(kx)\right]_0^{\pi}\right) \\ &= \frac{2}{k}(-1)^{k+1}. \end{aligned}$$

(4 points)

(b) Extending f to a 2π-periodic function leads to discontinuities at the points $x = (2k+1)\pi$ with $k \in \mathbb{Z}$. The general theorem about pointwise convergence states that at such points the Fourier series converges to the average of the left and right hand limits. In this case, we obtain

$$\frac{\lim_{x\to\pi^-} f(x) + \lim_{x\to-\pi^+} f(x)}{2} = \frac{2\pi + 0}{2} = \pi.$$

(3 points)

(c) The Fourier series does not converge uniformly on $\mathbb{R}$. Note that the partial sums are continuous functions and recall that uniform convergence preserves continuity. Since the limiting function is not continuous at odd multiples of π, we conclude that the convergence cannot be uniform.
(3 points)

Solution of problem 3 (15 points)

The educated guess $u(t,x) = e^{\lambda t}v(x)$ gives the following boundary value problem for v:

$$v''(x) - \lambda v(x) = 0, \quad v(0) = 0, \quad v'(1) - v(1) = 0.$$

(2 points)

The case $\lambda = -\omega^2 < 0$ gives

$$v(x) = a\cos(\omega x) + b\sin(\omega x).$$

The boundary condition at $x = 0$ implies that $a = 0$. The boundary condition at $x = 1$ implies that

$$b(\omega\cos(\omega) - \sin(\omega)) = 0.$$

The equation $\tan(x) = -x$ has countably many solutions $\omega_k > 0$. These give the nontrivial solutions

$$v_k(x) = \sin(\omega_k x), \quad k = 1, 2, 3, \ldots$$

(4 points)

The case $\lambda = 0$ gives $v(x) = a + bx$. The boundary conditions imply that $a = 0$ and that b is arbitrary. Setting $b = 1$ gives the nontrivial solution $v_0(x) = x$.
(2 points)

The case $\lambda = \omega^2 > 0$ gives

$$v(x) = a\cosh(\omega x) + b\sinh(\omega x)$$

The boundary condition at $x = 0$ implies that $a = 0$. The boundary condition at $x = 1$ implies that

$$b(\omega\cosh(\omega) - \sinh(\omega)) = 0.$$

For a nontrivial solution we need $\tanh(\omega) = \omega$ which has no positive solutions. Therefore, we do not get nontrivial solutions in this case.
(4 points)

Superposition of all nontrivial solutions gives the series expansion

$$u(t,x) = \sum_{k=0}^{\infty} c_k e^{-\lambda_k t} v_k(x) = c_0 x + \sum_{k=1}^{\infty} c_k e^{-\omega_k^2 t} \sin(\omega_k x).$$

(1 point)

Finally, we can use the initial condition and orthogonality relations to determine the coefficients:

$$c_0 = \frac{\int_0^1 x f(x)\,dx}{\int_0^1 x^2\,dx} \quad \text{and} \quad c_k = \frac{\int_0^1 f(x)\sin(\omega_k x)\,dx}{\int_0^1 \sin(\omega_k x)^2\,dx}.$$

(2 points)